

GENERALIZED HELMHOLTZ CONDITIONS FOR NON-CONSERVATIVE LAGRANGIAN SYSTEMS

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ABSTRACT. In this paper we provide generalized Helmholtz conditions, in terms of a semi-basic 1-form, which characterize when a given system of second order ordinary differential equations is equivalent to the Lagrange equations, for some given arbitrary non-conservative forces. For the particular cases of dissipative or gyroscopic forces, these conditions, when expressed in terms of a multiplier matrix, reduce to those obtained in [18]. When the involved geometric structures are homogeneous with respect to the fibre coordinates, we show how one can further simplify the generalized Helmholtz conditions. We provide examples where the proposed generalized Helmholtz conditions, expressed in terms of a semi-basic 1-form, can be integrated and the corresponding Lagrangian and Lagrange equations can be found.

1. INTRODUCTION

The classic inverse problem of Lagrangian mechanics requires to find the necessary and sufficient conditions, which are called *Helmholtz conditions*, such that a given system of second order ordinary differential equations (SODE) is equivalent to the Euler-Lagrange equations of some regular Lagrangian function. The problem has a long history and the literature about the subject is vast. There are various approaches to this problem, using different techniques and mathematical tools, [1, 3, 7, 9, 12, 17, 19, 24].

In this work we discuss the inverse problem of Lagrangian systems with non-conservative forces. Locally, the problem can be formulated as follows. We consider a SODE in normal form

$$(1.1) \quad \frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = 0$$

and an arbitrary covariant force field $\sigma_i(x, \dot{x})dx^i$. We provide necessary and sufficient conditions, which we call *generalized Helmholtz conditions*, for the existence of a Lagrangian L , such that the system (1.1) is equivalent to the Lagrange equations

$$(1.2) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = \sigma_i(x, \dot{x}).$$

When the covariant forces are of dissipative or gyroscopic type, the problem has been studied recently in [11, 18]. In these two papers the authors provide generalized Helmholtz conditions, in terms of a multiplier matrix, for a SODE (1.1) to represent Lagrange equations with non-conservative forces of dissipative or gyroscopic type.

The structure of the paper is as follows. In Section 2 we use the Frölicher-Nijenhuis formalism [13, 15] to provide a geometric framework associated to a given system (1.1). This framework includes: a nonlinear connection, dynamical covariant derivative and curvature type tensors. In

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Section 3 we use this geometric setting to reformulate the inverse problem of Lagrangian systems with non-conservative forces. The main contribution of this paper is to provide, in Theorem 3.2, generalized Helmholtz conditions in terms of semi-basic 1-forms for the most general case of the problem. Such semi-basic 1-form will represent the Poincaré-Cartan 1-form of the sought after Lagrangian. In the particular case when the covariant force field σ is zero, the generalized Helmholtz conditions $(GH_1) - (GH_3)$ of Theorem 3.2 reduce to, and simplify, the Helmholtz obtained in [3, Theorem 4.1].

In the next two sections we show that the proposed generalized Helmholtz conditions $(GH_1) - (GH_3)$, of Theorem 3.2, reduce to those obtained in [11, 18], which were expressed in terms of a multiplier matrix, for the particular case of dissipative or gyroscopic forces. Theorem 4.2 provides three equivalent sets of conditions, in terms of semi-basic 1-forms, for a SODE to be of dissipative type. One advantage of formulating the generalized Helmholtz conditions in terms of forms is discussed in Proposition 4.3, where we study the formal integrability of such conditions. An important consequence of Proposition 4.3 is that any SODE on a 2-dimensional manifold is of dissipative type. Theorem 5.2 provides two equivalent sets of generalized Helmholtz conditions, in terms of semi-basic 1-forms, which characterize Lagrangian systems of gyroscopic type.

In section 6 we discuss the inverse problem of Lagrangian systems with non-conservative forces, when all the involved geometric objects are homogeneous with respect to the velocity coordinates. Within this context, in Theorem 6.3, we prove that one generalized Helmholtz condition is a consequence of the other two. When the covariant force field is zero, depending on the degree of homogeneity, the problem reduces to the Finsler metrizable problem or the projective metrizable problem.

In the last section we show how the techniques developed throughout the paper can be used to discuss various examples. For these examples, the generalized Helmholtz conditions, expressed in terms of a semi-basic 1-form, can be integrated and therefore we can find the corresponding Lagrangian and Lagrange equations. The examples we analyse consist of non-variational projectively metrizable sprays that are of dissipative type and a class of gyroscopic semisprays. For each of the two examples, the techniques used in the proof of Theorems 4.2 and 5.2 are very useful for integrating the corresponding generalized Helmholtz conditions.

2. THE GEOMETRIC FRAMEWORK

2.1. A geometric setting for semisprays. In this section, we use the Frölicher-Nijenhuis formalism [13] to associate a geometric setting to a given system of second order ordinary differential equations, [2, 3, 14, 15].

For an n -dimensional smooth manifold M , denote by TM its tangent bundle. Local coordinates (x^i) on M induce local coordinates (x^i, y^i) on TM . The set smooth functions on M will be denoted by $C^\infty(M)$, while the set of smooth vector fields on M will be denoted by $\mathfrak{X}(M)$.

Consider $\mathbb{C} \in \mathfrak{X}(TM)$ the Liouville (dilation) vector field and J the tangent structure (vertical endomorphism). Throughout this paper we use the summation convention over covariant and contravariant repeated indices. With this convention, the Liouville vector field and the tangent structure are locally given by:

$$\mathbb{C} = y^i \frac{\partial}{\partial y^i}, \quad J = \frac{\partial}{\partial y^i} \otimes dx^i.$$

We consider the regular n -dimensional vertical distribution, $VTM : u \in TM \rightarrow V_u TM = \text{Ker } d_u \pi \subset T_u TM$. The forms dual to the vertical vector fields will play an important role in

this work. These are semi-basic (vector valued) forms on TM , with respect to the canonical projection π .

In order to develop a geometric setting, we will make use of the Frölicher-Nijenhuis formalism. Within this formalism one can identify derivations to vector valued forms on TM , [13, 15].

For a vector valued p -form P on TM , we denote by $i_P : \Lambda^k(TM) \rightarrow \Lambda^{k+p-1}(TM)$ the derivation of degree $(p-1)$, given by

$$i_P \alpha(X_1, \dots, X_{k+p-1}) = \frac{1}{p!(k-1)!} \sum_{\sigma \in S_{k+p-1}} \text{sign}(\sigma) \alpha(P(X_{\sigma(1)}, \dots, X_{\sigma(p)}), X_{\sigma(p+1)}, \dots, X_{\sigma(k+p-1)}),$$

where S_{k+p-1} is the permutation group of $\{1, \dots, k+p-1\}$. We denote by $d_P : \Lambda^k(TM) \rightarrow \Lambda^{k+p}(TM)$ the derivation of degree p , given by

$$d_P = [i_P, d] = i_P \circ d - (-1)^{p-1} d \circ i_P.$$

For two vector valued forms K and P on TM , of degrees k and p , we consider the Frölicher-Nijenhuis bracket $[K, P]$, which is the vector valued $(k+p)$ -form, uniquely determined by

$$(2.1) \quad d_{[K, P]} = [d_K, d_P] = d_K \circ d_P - (-1)^{kp} d_P \circ d_K.$$

For various commutation formulae, within the Frölicher-Nijenhuis formalism, we will use the Appendix A of the book [15]. We will use some vector valued forms on TM to associate a differential calculus on TM . Directly from the definition of the tangent structure J it follows that $[J, J] = 0$ and according to formula (2.1) it follow that $d_J^2 = 0$. Therefore, any d_J -exact form is d_J -closed and according to a Poincaré-type Lemma [26], any d_J -closed form is locally d_J -exact. The derivation d_J corresponds to the operator \hat{d} used in [16].

A *semispray*, or a second order vector field, is a globally defined vector field on TM , $S \in \mathfrak{X}(TM)$, that satisfies $JS = \mathbb{C}$. Locally, it can be expressed as:

$$(2.2) \quad S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}.$$

A curve $c : t \in I \subset \mathbb{R} \rightarrow c(t) = (x^i(t)) \in M$ is a *geodesic* of the semispray S if $c' : t \in I \subset \mathbb{R} \rightarrow c'(t) = (x^i(t), dx^i/dt) \in TM$, is an integral curve of S , which means that it satisfies (1.1).

A semispray S induces a horizontal and a vertical projector, h and v that are given by, [14],

$$h = \frac{1}{2} (\text{Id} - [S, J]), \quad v = \frac{1}{2} (\text{Id} + [S, J]).$$

Locally, the two projectors h and v can be expressed as

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N_j^i dx^j, \quad N_j^i = \frac{\partial G^i}{\partial y^j}.$$

The Frölicher-Nijenhuis bracket $[S, h]$ induces two geometric structures. One is the *almost complex structure*, \mathbb{F} , and the other one is the *Jacobi endomorphism*, Φ ,

$$(2.3) \quad \mathbb{F} = h \circ [S, h] - J, \quad \Phi = v \circ [S, h].$$

Locally, the almost complex structure can be expressed as follows

$$\mathbb{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i.$$

The Jacobi endomorphism has the following local expression

$$(2.4) \quad \Phi = R_j^i \frac{\partial}{\partial y^i} \otimes dx^j, \quad R_j^i = 2 \frac{\partial G^i}{\partial x^j} - S(N_j^i) - N_r^i N_j^r.$$

The horizontal distribution induced by a semispray is, in general, non-integrable. The obstruction to its integrability is given by the curvature tensor

$$(2.5) \quad R = \frac{1}{2}[h, h] = \frac{1}{2}R_{jk}^i \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \quad R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}.$$

The Jacobi endomorphism Φ and the curvature tensor R are closely related by

$$(2.6) \quad 3R = [J, \Phi], \quad 3R_{jk}^i = \frac{\partial R_j^i}{\partial y^k} - \frac{\partial R_k^i}{\partial y^j}.$$

From the above first formula and (2.1) it follows the commutation formula

$$(2.7) \quad [d_J, d_\Phi] = 3d_R.$$

We introduce now the *dynamical covariant derivative*, ∇ , associated to a semispray, following the approach from [2, 3]. For $f \in C^\infty(TM)$ and $X \in \mathfrak{X}(TM)$, we define

$$(2.8) \quad \nabla f = Sf, \quad \nabla X = h[S, hX] + v[S, vX].$$

Using the formulae (2.3), we can write the action of ∇ on vector fields as

$$(2.9) \quad \nabla = h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v = \mathcal{L}_S + \mathbb{F} + J - \Phi.$$

Therefore, the action of ∇ on the exterior algebra of TM is given by

$$(2.10) \quad \nabla = \mathcal{L}_S - i_{\mathbb{F}+J-\Phi}.$$

The following commutation formula can be shown using items iii) and iv) of [3, Theorem 3.5]

$$(2.11) \quad [d_J, \nabla] = d_h + 2i_R.$$

For more properties of the dynamical covariant derivative and some commutation formulae with other geometric structures, we refer to [3, Section 3.2].

2.2. Lagrange systems and non-conservative covariant forces. Consider $L : TM \rightarrow \mathbb{R}$ a Lagrangian, which is a smooth function on TM whose Hessian with respect to the fibre coordinates

$$(2.12) \quad g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}$$

is nontrivial. We say that L is a *regular Lagrangian* if the Poincaré-Cartan 2-form $dd_J L$ is a symplectic form on TM . Locally, the regularity condition of a Lagrangian L is equivalent to the fact that the Hessian (2.12) of L has maximal rank n on TM .

For an arbitrary semispray S and a Lagrangian L , the following 1-form (called the *Euler-Lagrange* 1-form, or *the Lagrange differential* in [25]) is a semi-basic 1-form:

$$(2.13) \quad \begin{aligned} \delta_S L &= \mathcal{L}_S d_J L - dL = d_J \mathcal{L}_S L - 2d_h L = \nabla d_J L - d_h L \\ &= \left\{ S \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} \right\} dx^i = \left\{ \frac{\partial S(L)}{\partial y^i} - 2 \frac{\delta L}{\delta x^i} \right\} dx^i = \left\{ \nabla \left(\frac{\partial L}{\partial y^i} \right) - \frac{\delta L}{\delta x^i} \right\} dx^i. \end{aligned}$$

The inverse problem of Lagrangian mechanics requires, for a given semispray S , to decide whether or not there exists a Lagrangian L with vanishing Lagrange differential, which means $\delta_S L = 0$. In this case we will say that the semispray S is *Lagrangian*. Locally it means that the solutions of the system (1.1) are among the solutions of the Euler-Lagrange equations of some Lagrangian L . Necessary and sufficient conditions for the existence of such Lagrangian are called Helmholtz conditions and were expressed in terms of a multiplier matrix [10, 12, 17, 24], a semi-basic 1-form [3, 10], or a 2-form [1, 9, 16, 19].

In this work we study the more general problem, when for a given semispray S and a semi-basic 1-form σ , we ask for the existence of a Lagrangian L , whose Lagrange differential is σ .

Definition 2.1. Consider S a semispray and $\sigma \in \Lambda^1(TM)$ a semi-basic 1 form. We say that S is of *Lagrangian type with covariant force field* σ if there exists a (locally defined) Lagrangian L such that $\delta_S L = \sigma$.

The above definition expresses the fact that the solutions of the system (1.1) are among the solutions of the Lagrange equations (1.2). If the Lagrangian L , which we search for, is regular, then the two systems (1.1) and (1.2) are equivalent.

There is an important aspect of Definition 2.1 that we want to emphasize, if we do not make any requirement about the covariant force field σ . For an arbitrary semispray S there is always a Lagrangian L and a semi-basic 1-form σ such that $\delta_S L = \sigma$. This case corresponds to the *semi-variational equations* studied in [23, Section 2]. In our analysis, we start with a given semispray S and a given semi-basic 1-form σ on TM and search for a Lagrangian L such that $\delta_S L = \sigma$. More exactly, for a given semispray S and a semi-basic 1-form σ , we provide necessary and sufficient conditions, which we will call *generalized Helmholtz conditions*, for the existence of a semi-basic 1-form θ that represents the Poincaré-Cartan 1-form of a Lagrangian L such that $\delta_S L = \sigma$.

3. GENERALIZED HELMHOLTZ CONDITIONS

In this section we provide necessary and sufficient conditions for a given semispray S to be of Lagrangian type with a given covariant force field σ . These conditions, which we will refer to as *generalized Helmholtz conditions*, will be expressed in terms of a semi-basic 1-form. We will prove that for some particular cases of the covariant force field (dissipative and gyroscopic) the generalized Helmholtz conditions reduce to those obtained in [18] in terms of a multiplier matrix.

Throughout this work, we make the following assumption about the semi-basic 1-form θ that will be involved in expressing the generalized Helmholtz conditions. We say that a semi-basic 1-form $\theta = \theta_i(x, y)dx^i$ is *non-trivial* if the matrix $g_{ij} = \partial\theta_i/\partial y^j$ is non-trivial. If $\theta = d_J L$ is the Poincaré-Cartan 1-form of some function L , then θ is non-trivial if and only if L is a Lagrangian.

Theorem 3.1. Consider S a semispray and $\sigma \in \Lambda^1(TM)$ a semi-basic 1-form. The semispray S is of Lagrangian type with covariant force field σ if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(TM)$ such that $\mathcal{L}_S \theta - \sigma$ is a closed 1-form on TM .

Proof. For the direct implication, from Definition 2.1, it follows that there exists a Lagrangian L such that $\mathcal{L}_S d_J L - dL = \sigma$. We take $\theta = d_J L$, the Poincaré-Cartan 1-form of L . It follows that $\mathcal{L}_S \theta - \sigma = dL$, which is exact and hence it is a closed 1-form. Since L is a Lagrangian, we have that the semi-basic 1-form θ is non-trivial.

For the converse implication, we assume that there exists $\theta \in \Lambda^1(TM)$ a non-trivial, semi-basic 1-form, such that $\mathcal{L}_S \theta - \sigma$ is a closed 1-form on TM . Therefore, there exists a (locally defined) function L on TM such that

$$(3.1) \quad \mathcal{L}_S \theta - \sigma = dL.$$

We apply i_J to both sides of this formula. In the right hand side we have $i_J dL = d_J L$. We evaluate now the left hand side. Since θ and σ are semi-basic 1-forms, it follows that $i_J \theta = i_J \sigma = 0$. For a vector valued 1-form K on TM , we use the commutation formula, see [15, A.1, page 205],

$$(3.2) \quad i_K \mathcal{L}_S = \mathcal{L}_S i_K + i_{[K, S]}.$$

If $K = J$, the tangent structure, we use above formula and $[J, S] = h - v$. It follows that $i_J \mathcal{L}_S \theta = i_{[J, S]} \theta = i_h \theta = \theta$ and hence $\theta = d_J L$. Therefore, the non-trivial, semi-basic 1 form θ is

the Poincaré-Cartan 1-form of L , and hence L is a Lagrangian function. We replace this in formula (3.1) and obtain that the semispray S is of Lagrangian type with the covariant force field σ . \square

In Theorem 3.1, if we search for a regular Lagrangian L , then the corresponding semi-basic 1-form θ has to be regular as well, in the following sense. A semi-basic 1-form $\theta \in \Lambda^1(TM)$ is said to be *regular* if $d\theta$ is a symplectic 2-form. If $\theta = d_J L$ is the Poincaré-Cartan 1-form of some function L , then the regularity condition of θ is equivalent to the regularity of the Lagrangian L .

Next theorem provides necessary and sufficient conditions for the existence of the semi-basic 1-form, which was discussed in Theorem 3.1, using the differential operators associated to a given semispray.

Theorem 3.2. *A semispray S is of Lagrangian type with covariant force field σ if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(TM)$ such that the following generalized Helmholtz conditions are satisfied:*

$$\begin{aligned} (GH_1) \quad & d_J \theta = 0; \\ (GH_2) \quad & d_\Phi \theta = \frac{1}{2} \nabla d_J \sigma - d_h \sigma; \\ (GH_3) \quad & \nabla d_v \theta = d_v \sigma - \frac{1}{2} i_{\mathbb{F}+J} d_J \sigma. \end{aligned}$$

Proof. We fix a semispray S and a semi-basic 1-form $\sigma \in \Lambda^1(TM)$. According to Theorem 3.1 we have that S is of Lagrangian type with covariant force field σ if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(TM)$ such that

$$(3.3) \quad \mathcal{L}_S d\theta = d\sigma.$$

We will prove now that formula (3.3) is equivalent to the three generalized Helmholtz conditions $(GH_1) - (GH_3)$.

For the direct implication, we consider θ a non-trivial, semi-basic 1-form on TM that satisfies formula (3.3). We apply to both sides of this formula the derivation i_J . Using commutation formula (3.2) for $K = J$ and the fact that $i_{[J,S]} d\theta = i_{h-v} d\theta = i_{2h-\text{Id}} d\theta = 2i_h d\theta - 2d\theta = 2d_h \theta$, we obtain

$$(3.4) \quad \mathcal{L}_S d_J \theta + 2d_h \theta = d_J \sigma.$$

We apply again the derivation i_J to both sides of the above formula and we use that $d_h \theta$ and $d_J \sigma$ are semi-basic 2-forms, which implies that $i_J d_h \theta = 0$ and $i_J d_J \sigma = 0$. Using again the commutation formula for i_J and \mathcal{L}_S , it follows that $i_{[J,S]} d_J \theta = 0$. Since $i_{[J,S]} d_J \theta = i_{h-v} d_J \theta = 2d_J \theta$ we obtain that $d_J \theta = 0$, which is the first generalized Helmholtz condition (GH_1) . We replace this in formula (3.4) and obtain the formula

$$(3.5) \quad d_h \theta = \frac{1}{2} d_J \sigma.$$

Since θ is a semi-basic 1-form, it follows that $i_h \theta = \theta$ and $i_v \theta = 0$. Therefore we have

$$d\theta = 2d\theta - d\theta = i_{\text{Id}} d\theta - d\theta = i_h d\theta + i_v d\theta - d\theta = d_h \theta + d_v \theta.$$

The condition $d_J \theta = 0$ reads $d_J \theta(X, Y) = d\theta(JX, Y) + d\theta(X, JY) = 0$, for all $X, Y \in \mathfrak{X}(TM)$. Hence, for any two vertical vector fields V, W on TM , we have $d\theta(V, W) = 0$.

In order to show that the next two generalized Helmholtz conditions are satisfied, we will prove first that

$$(3.6) \quad i_{\mathbb{F}+J} d_v \theta = 0.$$

Consider $X_1, Y_1 \in \mathfrak{X}(TM)$. There exist $X_2, Y_2 \in \mathfrak{X}(TM)$ such that $(\mathbb{F} + J)(X_1) = hX_2$ and $(\mathbb{F} + J)(Y_1) = hY_2$. Moreover, if we compose to the left these two equalities with the tangent

structure J and use the fact that $J \circ \mathbb{F} = v$ and $J \circ h = J$, we obtain $vX_1 = JX_2$ and $vY_1 = JY_2$. Using these equalities we have

$$\begin{aligned} (i_{\mathbb{F}+J}d_v\theta)(X_1, Y_1) &= d\theta((\mathbb{F}+J)(X_1), vY_1) + d\theta(vX_1, (\mathbb{F}+J)(Y_1)) \\ &= d\theta(hX_2, JY_2) + d\theta(JX_2, hY_2) = d_J\theta(X_2, Y_2) = 0. \end{aligned}$$

We apply the derivation i_h to both sides of formula (3.3) and use the commutation rule (3.2) for $K = h$, which gives

$$\mathcal{L}_S i_h d\theta + i_{[h,S]} d\theta = i_h d\sigma.$$

We use the fact that θ and σ are semi-basic forms, which implies that $i_h d\theta = d_h\theta - d\theta$ and $i_h d\sigma = d_h\sigma - d\sigma$, and formula (3.3) again to obtain

$$(3.7) \quad \mathcal{L}_S d_h\theta + i_{[h,S]} d\theta = d_h\sigma.$$

From the two formulae (2.3) we have $[h, S] = -\mathbb{F} - J - \Phi$. Now using formula (3.6) we obtain $i_{[h,S]} d\theta = -i_{\mathbb{F}+J} d\theta - i_\Phi d\theta = -i_{\mathbb{F}+J} d_h\theta - d_\Phi\theta$. With this formula we go back to (3.7), where we use the fact that $\mathcal{L}_S d_h\theta - i_{\mathbb{F}+J} d_h\theta = \nabla d_h\theta$ and hence

$$(3.8) \quad \nabla d_h\theta - d_\Phi\theta = d_h\sigma.$$

If we make use of formula (3.5) to substitute $d_h\theta$, we obtain that the second generalized Helmholtz condition (GH_2) is true as well. Using formula (2.10) we obtain

$$\mathcal{L}_S d\theta = \nabla d\theta + i_{\mathbb{F}+J} d\theta - i_\Phi d\theta.$$

If we replace this in (3.3), we obtain

$$\nabla d_h\theta + \nabla d_v\theta + i_{\mathbb{F}+J} d_h\theta - d_\Phi\theta = d_h\sigma + d_v\sigma.$$

In the above formula we use (3.8) and formula (3.5) and obtain that the last generalized Helmholtz condition (GH_3) is true as well.

We will prove now the converse, which means that the three generalized Helmholtz conditions $(GH_1) - (GH_3)$ imply the condition (3.3). We will prove first that the existence of a non-trivial, semi-basic 1-form θ that satisfies the three conditions $(GH_1) - (GH_3)$ implies the existence of a non-trivial, semi-basic 1-form $\tilde{\theta}$ that satisfies $(GH_1) - (GH_3)$ and (3.5) as well.

Consider θ a non-trivial, semi-basic 1-form that satisfies the generalized Helmholtz conditions $(GH_1) - (GH_3)$. We apply the derivation d_J to both sides of formula (GH_2) to obtain

$$(3.9) \quad d_J d_\Phi\theta = \frac{1}{2} d_J \nabla d_J\sigma - d_J d_h\sigma.$$

We evaluate first each of the two sides of the above formula. For the left hand side, using the commutation formula (2.7), as well as the fact that $d_J\theta = 0$, we have

$$d_J d_\Phi\theta = d_{[J,\Phi]}\theta = 3d_R\theta = 3d_h d_h\theta.$$

Using the commutation formula (2.11), the fact that $d_J^2 = 0$ and the fact that $i_R d_J\sigma = 0$, we can express the first term of the right hand side of formula (3.9) as

$$d_J \nabla d_J\sigma = d_h d_J\sigma.$$

Since $[J, h] = 0$ it follows that $d_h d_J + d_J d_h = 0$. Now, if we replace everything in both sides of formula (3.9) we obtain $3d_h d_h\theta = 3d_h d_J\sigma/2$, which can be further written as

$$(3.10) \quad d_h \left(d_h\theta - \frac{1}{2} d_J\sigma \right) = 0.$$

If we apply the derivation d_J to both sides of formula (GH_3) we obtain

$$(3.11) \quad d_J \nabla d_v \theta = d_J d_v \sigma - \frac{1}{2} d_J i_{\mathbb{F}+J} d_J \sigma.$$

To evaluate the left hand side of formula (3.11) we use the commutation formula (2.11)

$$d_J \nabla d_v \theta = \nabla d_J d_v \theta + d_h d_v \theta + 2i_R d_v \theta.$$

We use the fact that $[J, v] = 0$, which implies that $d_J d_v \theta + d_v d_J \theta = 0$, to obtain that $d_J d_v \theta = 0$. Since $2R = [h, h] = [h, \text{Id} - v] = -[h, v]$ it follows that

$$d_h d_v \theta + d_v d_h \theta = d_{[h, v]} = -2d_R \theta = -2i_R d_v \theta.$$

We use all these calculations to write the left hand side of formula (3.11) as

$$d_J \nabla d_v \theta = -d_v d_h \theta.$$

Finally, we have to evaluate the right hand side of formula (3.11). For its second term, we will use the following commutation formula, [15, A.1, page 205], for two vector valued 1-forms K and P

$$(3.12) \quad i_K d_P = d_P i_K + d_{P \circ K} - i_{[K, P]}.$$

For $P = J$ and $K = \mathbb{F} + J$ we have

$$i_{\mathbb{F}+J} d_J d_J \sigma = d_J i_{\mathbb{F}+J} d_J \sigma + d_{J \circ (\mathbb{F}+J)} d_J \sigma - i_{[\mathbb{F}+J, J]} d_J \sigma.$$

Since $J \circ (\mathbb{F} + J) = v$, $[\mathbb{F} + J, J] = [\mathbb{F}, J] = -R$ and $i_{[\mathbb{F}+J, J]} d_J \sigma = -i_R d_J \sigma = 0$ we obtain

$$d_J i_{\mathbb{F}+J} d_J \sigma = -d_v d_J \sigma.$$

Using these calculations, we can write the right hand side of formula (3.11) as

$$d_J d_v \sigma - \frac{1}{2} d_J i_{\mathbb{F}+J} d_J \sigma = -d_v d_J \sigma + \frac{1}{2} d_v d_J \sigma = -\frac{1}{2} d_v d_J \sigma$$

It follows that one can write formula (3.11) as

$$(3.13) \quad d_v \left(d_h \theta - \frac{1}{2} d_J \sigma \right) = 0.$$

From the two formulae (3.10) and (3.13), we obtain

$$(3.14) \quad d \left(d_h \theta - \frac{1}{2} d_J \sigma \right) = 0.$$

Using the above formula, there exists a locally defined basic 1-form β such that

$$(3.15) \quad d_h \theta - \frac{1}{2} d_J \sigma = d\beta.$$

The semi-basic 1-form $\tilde{\theta} = \theta - \beta$ satisfies all three generalized Helmholtz condition $(GH_1) - (GH_3)$ and formula (3.5) as well. Since β is a basic 1-form, we have that the semi-basic 1-form $\tilde{\theta}$ is non-trivial if and only if the semi-basic 1-form θ is non-trivial.

For this non-trivial, semi-basic 1-form $\tilde{\theta}$, we will prove that formula (3.3) is true. Formula (3.6), which we proved in the first part of our proof, is still true for $\tilde{\theta}$ since for this we only need that $\tilde{\theta}$ is a semi-basic 1-form that satisfies $d_J \tilde{\theta} = 0$. Using formula (2.10) we obtain

$$(3.16) \quad \mathcal{L}_S d\tilde{\theta} = \nabla d_h \tilde{\theta} + \nabla d_v \tilde{\theta} + i_{\mathbb{F}+J} d_h \tilde{\theta} - d_\Phi \tilde{\theta}.$$

In the right hand side of formula (3.16) we replace $d_h \tilde{\theta}$, $\nabla d_v \tilde{\theta}$ and $d_\Phi \tilde{\theta}$ in terms of σ , from (3.5) and the conditions $(GH_2) - (GH_3)$. It follows that formula (3.3) is true. \square

In the absence of the exterior force, which means that $\sigma = 0$, the generalized Helmholtz conditions of Theorem 3.2 reduce to the Helmholtz conditions in [3, Theorem 4.1].

Corollary 3.3. *A semispray S is Lagrangian if and only if there exists a non-trivial, semi-basic 1-form θ that satisfies the following Helmholtz conditions*

$$\begin{aligned} (H_1) \quad & d_J\theta = 0; \\ (H_2) \quad & d_\Phi\theta = 0; \\ (H_3) \quad & \nabla d_v\theta = 0. \end{aligned}$$

In [3, Theorem 4.1] there is an extra condition $d_h\theta = 0$ that has been used. As we have seen in the proof of the second part of Theorem 3.2, it can be shown that the three Helmholtz conditions $(H_1) - (H_3)$ imply that $d_h\theta = d\beta$, for a basic 1-form β . Therefore, the new semi-basic 1-form $\tilde{\theta} = \theta - \beta$ satisfies the three Helmholtz conditions $(H_1) - (H_3)$ as well as the fourth condition $d_h\tilde{\theta} = 0$, which was used in [3, Theorem 4.1].

We will provide now a local description of the three generalized Helmholtz conditions $(GH_1) - (GH_3)$. Consider S a semispray, locally given by formula (2.2), and let $\theta = \theta_i dx^i$, $\sigma = \sigma_i dx^i$ be two semi-basic 1-forms on TM . We have

$$\begin{aligned} d_v\theta &= g_{ij}\delta y^j \wedge dx^i, \quad g_{ij} := \frac{\partial \theta_i}{\partial y^j}. \\ d_J\theta &= \frac{1}{2}(g_{ij} - g_{ji})dx^j \wedge dx^i, \quad d_\Phi\theta = \frac{1}{2}(g_{ik}R_j^k - g_{jk}R_i^k)dx^j \wedge dx^i. \end{aligned}$$

The generalized Helmholtz conditions $(GH_1) - (GH_3)$ can be expressed locally as follows

$$\begin{aligned} (LGH_1) \quad & g_{ij} = g_{ji}. \text{ Due to the above definition for } g_{ij}, \text{ we also have } \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}. \\ (LGH_2) \quad & g_{ik}R_j^k - g_{jk}R_i^k = \frac{1}{2}\nabla \left(\frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right) - \left(\frac{\delta \sigma_i}{\delta x^j} - \frac{\delta \sigma_j}{\delta x^i} \right). \\ (LGH_3) \quad & \nabla g_{ij} = \frac{1}{2} \left(\frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right). \end{aligned}$$

Condition (LGH_3) and the local expression of $d_h\theta = 0$ appear also in [4, Theorem 3.1], as conditions that uniquely fix the nonlinear connection of a Lagrange space and a non-conservative force. If $\sigma = 0$, the conditions $(LGH_1) - (LGH_3)$ represent the classic Helmholtz conditions in terms of the multiplier matrix g_{ij} .

4. THE DISSIPATIVE CASE

In this section we restrict our results from the previous section to the particular case when the covariant force field is a d_J -closed, semi-basic 1-form σ . Such systems were studied in [22], with special attention on a subclass that admits a Lagrangian description.

Definition 4.1. A semispray S is said to be of *dissipative type* if there exists a Lagrangian L and a d_J -closed semi-basic 1-form σ on TM such that $\delta_S L = \sigma$.

This definition includes the classic *dissipation of Rayleigh type*, where $\sigma = d_J\mathcal{D}$, for \mathcal{D} a negative definite, quadratic function in the velocities.

Next theorem provides three sets of equivalent conditions that characterize dissipative systems in terms of semi-basic 1-forms, and it corresponds to the multiplier matrix characterisations from [18, Corollary 1, Theorem 1, Theorem 3]. The techniques we employ in the proof of Theorem 4.2 can be used to integrate the corresponding generalized Helmholtz conditions, as it will be shown in the example of Subsection 7.1.

Two of the three equivalent sets of conditions of Theorem 4.2, (D_1) and (D_3) , do not involve the dissipative force field σ , while the other set, (D_2) , does. The proof of the theorem shows how to recover the covariant force field σ , when it is not given.

Theorem 4.2. *A semispray S is of dissipative type if and only if there exist a non-trivial, semi-basic 1-form θ and a d_J -closed semi-basic 1-form σ on TM that satisfy one of the following equivalent sets of conditions*

$$\begin{aligned} (D_1) \quad & d_J\theta = 0, \quad d_h\theta = 0; \\ (D_2) \quad & d_J\theta = 0, \quad d_\Phi\theta = -d_h\sigma, \quad \nabla d_v\theta = d_v\sigma; \\ (D_3) \quad & d_J\theta = 0, \quad d_R\theta = 0, \quad d_h d_v\theta = 0. \end{aligned}$$

Proof. We will prove the following implications: $(D_1) \implies S \text{ is dissipative} \implies (D_2) \implies (D_3) \implies (D_1)$.

For the *first implication*, we assume that there exists a non-trivial, semi-basic 1-form θ such that $d_J\theta = 0$ and $d_h\theta = 0$. From the first condition (D_1) , it follows that there exists a Lagrangian L , locally defined on TM , such that $\theta = d_JL$. Now, the second condition (D_1) reads $0 = d_h d_JL = -d_J d_hL$. This means that the semi-basic 1-form d_hL is d_J -closed and hence it is locally d_J -exact. Therefore, there exists a function f , locally defined on TM , such that $d_hL = d_Jf$. Consider now the function $\mathcal{D} = S(L) - 2f$. It follows that $d_J S(L) - 2d_hL = d_J\mathcal{D}$. In view of the second expression (2.13) of $\delta_S L$, it follows that $\delta_S L = d_J\mathcal{D}$, which shows that the semispray S is of dissipative type, with the d_J -exact (and hence d_J -closed) semi-basic 1-form $\sigma = d_J\mathcal{D}$.

For the *second implication*, we assume that the semispray S is of dissipative type. By definition, it follows that there exist a Lagrangian L and a d_J -closed semi-basic 1-form σ on TM such that $\delta_S L = \sigma$. Last condition implies that formula (3.3) is true, where $\theta = d_JL$ and $d_J\sigma = 0$. Therefore the three generalized Helmholtz conditions $(GH_1) - (GH_3)$ of Theorem 3.2 are satisfied. If we use the fact that $d_J\sigma = 0$, we have that the conditions $(GH_1) - (GH_3)$ imply the conditions (D_2) .

For the *third implication*, we consider θ a non-trivial, semi-basic 1-form and σ a d_J -closed semi-basic 1-form on TM such that the three conditions (D_2) are satisfied.

Using formulae (2.1) and (2.6), as well as the first two conditions (D_2) , it follows

$$(4.1) \quad 3d_R\theta = d_{[J,\Phi]}\theta = d_J d_\Phi\theta + d_\Phi d_J\theta = -d_J d_h\sigma = d_h d_J\sigma = 0.$$

We apply the derivation d_J to both sides of the last condition (D_2) . Since $[J, v] = 0$, using formula (2.1), it follows that $d_J d_v\sigma = -d_v d_J\sigma = 0$. Using the commutation formula (2.11), we have

$$0 = d_J d_v\sigma = d_J \nabla d_v\theta = \nabla d_J d_v\theta + d_h d_v\theta + 2i_R d_v\theta.$$

In the above formula we use that $i_R d_v\theta = i_R d\theta = d_R\theta = 0$ and $d_J d_v\theta = 0$. It follows that $d_h d_v\theta = 0$.

For the *last implication*, we show first that $d_h\theta$ is a closed, basic 2-form. The second condition (D_3) , $d_R\theta = 0$, is equivalent to $d_h d_h\theta = 0$. Using (2.1), we have $d_h d_v\theta + d_v d_h\theta = d_{[h,v]}\theta = -2d_R\theta = 0$. In this last formula we use the last condition (D_3) and obtain $d_v d_h\theta = 0$. Since $d_h\theta$ is a semi-basic 2-form it follows that

$$(4.2) \quad dd_h\theta = d_h d_h\theta + d_v d_h\theta = 0.$$

Above formula implies that $d_h\theta$ is a closed, basic 2-form and hence it is locally exact. Therefore, there exists a basic 1-form η such that $d_h\theta = d\eta$. We consider now the semi-basic 1-form

$$(4.3) \quad \tilde{\theta} = \theta - \eta.$$

Since η is a basic 1-form, we have that the semi-basic 1-forms $\tilde{\theta}$ and θ are simultaneously non-trivial. We have that $d_J\tilde{\theta} = d_J\theta = 0$ and $d_h\tilde{\theta} = 0$, which means that the non-trivial, semi-basic 1-form $\tilde{\theta}$ satisfies the two conditions (D_1) . \square

The three conditions (D_2) are equivalent to the four conditions of [18, Theorem 1]. The correspondence between the semi-basic 1-form $\theta = \theta_i dx^i$ and the $(0, 2)$ -type tensor $g = (g_{ij})$ is $g_{ij} = \partial\theta_i/\partial y^j$. The three conditions (D_3) are equivalent to the four conditions of [18, Theorem 3].

Locally, the three equivalent sets of conditions (D_1) , (D_2) and (D_3) can be expressed as follows

$$(LD_1) \quad \frac{\partial\theta_i}{\partial y^j} = \frac{\partial\theta_j}{\partial y^i}, \quad \frac{\delta\theta_i}{\delta x^j} = \frac{\delta\theta_j}{\delta x^i}.$$

$$(LD_2) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \quad g_{ik}R_j^k - g_{jk}R_i^k = \frac{\delta}{\delta x^i} \left(\frac{\partial \mathcal{D}}{\partial y^j} \right) - \frac{\delta}{\delta x^j} \left(\frac{\partial \mathcal{D}}{\partial y^i} \right), \quad \nabla g_{ij} = \frac{\partial^2 \mathcal{D}}{\partial y^i \partial y^j}.$$

$$(LD_3) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \quad g_{il}R_{jk}^l + g_{kl}R_{ij}^l + g_{jl}R_{ki}^l = 0, \quad g_{ij|k} - g_{ik|j} = 0.$$

In the last set of conditions (LD_3) , $g_{ij|k} = \delta g_{ij}/\delta x^k - g_{il}\Gamma_{jk}^l - g_{lj}\Gamma_{ik}^l$, where $\Gamma_{jk}^i = \partial N_j^i/\partial y^k$, is the h -covariant derivative of the tensor g with respect to the Berwald connection. The conditions (LD_3) represent conditions (45)-(46) in [18], while (LD_2) represent conditions (30)-(32) in [18].

For the formal integrability of the system (D_1) : $d_J\theta = 0$ and $d_h\theta = 0$, we can follow the results of [5, 8], where the formal integrability of very similar systems was investigated. Using a very similar proof as the one of Theorems 4.2 and 4.3 in [5] and Theorems 3 and 4 in [8], we can state the following result.

Proposition 4.3. *The system (D_1) is formally integrable if the following obstruction is satisfied for all semi-basic 1-forms θ*

$$(4.4) \quad d_R\theta = 0.$$

An important consequence of the previous proposition appears in dimension 2. In this case, the obstruction $d_R\theta = 0$ is automatically satisfied, being a semi-basic 3-form on a 2-dimensional manifold. Therefore, we obtain the following corollary.

Corollary 4.4. *Any semispray on a 2-dimensional manifold is of dissipative type.*

The first example of Section 5 in [18], as well as the example (7.2) that we discuss in Section 7 agree with the conclusion of the above corollary.

5. THE GYROSCOPIC CASE

In this section, we restrict the general results obtained in Section 3 to the particular case when the exterior covariant force field σ is of the type $i_S\omega$, for ω a basic 2-form. This corresponds to the gyroscopic case, studied in [18].

Definition 5.1. A semispray S is said to be of *gyroscopic type* if there exists a Lagrangian L and a basic 2-form ω on TM such that $\delta_S L = i_S\omega$.

Next theorem provides two characterisations of gyroscopic systems in terms of a semi-basic 1-form, and it corresponds to the multiplier matrix characterisations from [18, Theorem 2, Theorem 4]. The techniques we employ in the proof of Theorem 5.2 can be used when studying the integrability of the corresponding generalized Helmholtz conditions, as we exemplify it in Subsection 7.2.

The set of conditions (G_2) , of Theorem 5.2, does not involve the gyroscopic 2-form ω , while the other set (G_1) does. The proof of the theorem shows how to recover this gyroscopic 2-form ω .

Theorem 5.2. *A semispray S is of gyroscopic type if and only if there exist a non-trivial, semi-basic 1-form θ and a basic 2-form ω on TM that satisfy one of the following equivalent sets of conditions*

$$(G_1) \quad d_J\theta = 0, \quad d_\Phi\theta = i_S d\omega, \quad \nabla d_v\theta = 0;$$

$$(G_2) \quad d_J\theta = 0, \quad d_\Phi\theta = i_S d_R\theta, \quad \nabla d_v\theta = 0.$$

Proof. We will prove the following implications: S is gyroscopic $\implies (G_1) \implies (G_2) \implies S$ is gyroscopic.

For the *first implication*, we assume that S is of gyroscopic type and therefore there exists a Lagrangian L and a basic 2-form ω such that $\delta_S L = i_S \omega$. Using third form of $\delta_S L$ in formula (2.13), we can write the condition that S is of gyroscopic type as follows

$$\nabla d_J L - d_h L = i_S \omega.$$

If we apply the derivation d_J to the both sides of the above formula, use the commutation formula (2.11), as well as the commutation $d_h d_J = -d_J d_h$, we obtain

$$(5.1) \quad 2d_h d_J L + 2i_R d_J L = d_J i_S \omega.$$

We evaluate the right hand side of (5.1) by using the commutation formula, [15, A.1, page 205],

$$(5.2) \quad i_X d_K = -d_K i_X + \mathcal{L}_{KX} + i_{[K,X]}.$$

For $K = J$ and $X = S$, this commutation formula reads

$$(5.3) \quad d_J i_S \omega = -i_S d_J \omega + \mathcal{L}_{\mathbb{C}} \omega + i_{[J,S]} \omega = 2\omega.$$

We consider the non-trivial, semi-basic 1-form $\theta = d_J L$, and use the fact that $i_R d_J L = 0$. Then, from formula (5.1) it follows $d_h \theta = \omega$ and hence $\mathcal{L}_S d_h \theta = \mathcal{L}_S \omega$. The condition $\delta_S L = i_S \omega$ implies $\mathcal{L}_S d\theta = di_S \omega$, which can be further written as

$$\mathcal{L}_S d_h \theta + \mathcal{L}_S d_v \theta = \mathcal{L}_S \omega - i_S d\omega \iff \mathcal{L}_S d_v \theta = -i_S d\omega.$$

In the last formula above, we use (2.10), (3.6) and $i_\Phi d_v \theta = d_\Phi \theta$ to obtain

$$\nabla d_v \theta - d_\Phi \theta = -i_S d\omega.$$

In both sides of this formula we separate the semi-basic 2-forms and obtain $d_\Phi \theta = i_S d\omega$, and what remains is $\nabla d_v \theta = 0$. Therefore, we have shown that all three conditions (G_1) are satisfied.

For the *second implication*, we apply the derivation d_J to both sides of the second condition (G_1) . Using the commutation formulae (2.7) and (5.2) it follows

$$3d_R \theta = d_{[J,\Phi]} \theta = d_J d_\Phi \theta = d_J i_S d\omega = i_{[J,S]} d\omega = 3d\omega.$$

This implies $d_R \theta = d\omega$ and hence the second condition (G_2) is satisfied.

For the *last implication*, we assume now that there is a non-trivial, semi-basic 1-form θ that satisfies the set of conditions (G_2) . We apply the derivation d_J to both sides of the last condition (G_2) and use the commutation formula (2.11), which implies

$$0 = d_J \nabla d_v \theta = d_h d_v \theta + 2i_R d_v \theta = d_v d_h \theta.$$

Last formula expresses the fact that $d_h \theta$ is a basic 2-form. We denote it $d_h \theta = \omega$ and have that

$$d\omega = dd_h \theta = d_h^2 \theta = d_R \theta.$$

From the first condition (G_2) , and the fact that the semi-basic 1-form θ is non-trivial, it follows that there exists a locally defined Lagrangian L such that $\theta = d_J L$. We have now $d_h d_J L = \omega$. Since ω is a basic 2-form, it follows that $2\omega = d_J i_S \omega$ and hence we obtain

$$-d_J d_h L = \frac{1}{2} d_J i_S \omega \iff d_J (2d_h L + i_S \omega) = 0.$$

From the last formula it follows that there is a locally defined function f on TM such that $2d_h L + i_S \omega = d_J f$. We consider now the function $g = S(L) - f$ and obtain

$$(5.4) \quad \delta_S L = d_J S(L) - 2d_h L = i_S \omega + d_J g.$$

From formula (5.4) it follows

$$(5.5) \quad \mathcal{L}_S d\theta = di_S d_h \theta + dd_J g,$$

which can be written further as

$$\mathcal{L}_S d_h \theta + \mathcal{L}_S d_v \theta = \mathcal{L}_S d_h \theta - i_S d_R \theta + dd_J g.$$

Using the last two conditions of the set (G_2) and the above formula, it follows that $dd_J g = 0$. We use this in formula (5.5) and obtain $\mathcal{L}_S d\theta = di_S \omega$, which in view of Theorem 3.1 gives $\delta_S L = i_S \omega$. Last formula shows that the spray S is of gyroscopic type. \square

The three conditions (G_1) , in terms of a semi-basic 1-form, are equivalent to the four conditions of [18, Theorem 2], expressed in terms of a multiplier matrix. The three conditions (G_2) are equivalent to the four conditions of [18, Theorem 4].

Locally, the two sets of equivalent conditions (G_1) and (G_2) can be written as follows.

$$(LG_1) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \quad g_{ik} R_j^k - g_{jk} R_i^k = \left(\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} \right) y^k, \quad \nabla g_{ij} = 0,$$

$$(LG_2) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \quad g_{ik} R_j^k - g_{jk} R_i^k = \left(g_{il} R_{jk}^l + g_{kl} R_{ij}^l + g_{jl} R_{ki}^l \right) y^k, \quad \nabla g_{ij} = 0.$$

Last two conditions (LG_1) represent conditions [18, (38)-(39)], while second condition (LG_2) represents condition [18, (49)].

6. THE HOMOGENEOUS CASE

In this section we study the case when all the involved geometric objects are homogeneous with respect to the fibre coordinates. Consequently, we will restrict all our geometric structures to the slit tangent bundle $T_0 M = TM \setminus \{0\}$.

A semispray $S \in \mathfrak{X}(T_0 M)$ is called a *spray* if it is 2-homogeneous, which means that $[\mathbb{C}, S] = S$. The coefficients $G^i(x, y)$, in formula (2.2), which are locally defined functions on $T_0 M$, are homogeneous of order 2 in the fibre coordinates.

Definition 6.1. Consider S a spray and $\sigma \in \Lambda^1(T_0 M)$ a semi-basic 1-form, homogeneous of order $p \in \mathbb{N}^*$. We say that the spray S is of *Finslerian type with covariant force field* σ if there exists a Lagrangian $L \in C^\infty(T_0 M)$, homogeneous of order p , such that $\delta_S L = \sigma$.

We will discuss now some particular cases of Definition 6.1 in the Finslerian context. For this, we recall the notion of a Finsler function, Finsler metrizable and projective metrizable.

Definition 6.2. A positive function $F : TM \rightarrow \mathbb{R}$ is called a *Finsler function* if it satisfies

- i) F is smooth on $T_0 M$ and continuous on the null section;
- ii) F is positively homogeneous with respect to the fibre coordinates: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda \geq 0$;

iii) F^2 is a regular Lagrangian on T_0M .

When $\sigma = 0$, depending on the values of p , we obtain two important particular cases of the inverse problem of Lagrangian mechanics. The case $\sigma = 0$, $p = 1$, and $L = F$, for a Finsler function F , is known as the *projective metrization problem* [5, 6, 15, 16]. The case $\sigma = 0$, $p = 2$, and $L = F^2$, for a Finsler function F , is known as the *Finsler metrization problem* [20]. For the general case, when $\sigma \neq 0$, we will see that we also have to make distinction between the two cases $p = 1$ and $p > 1$.

6.1. The case $p > 1$. Next theorem shows that, in the homogeneous context, the generalized Helmholtz condition (GH_2) is a consequence of the other two generalized Helmholtz conditions (GH_1) and (GH_3) of Theorem 3.2.

Theorem 6.3. *Consider S a spray and $\sigma \in \Lambda^1(T_0M)$ a semi-basic 1-form, homogeneous of order $p > 1$. The spray S is of Finslerian type with covariant force field σ if and only if there exists a non-trivial, semi-basic form $\theta \in \Lambda^1(T_0M)$, homogeneous of order $(p - 1)$ that satisfies the two generalized Helmholtz conditions (GH_1) and (GH_3) of Theorem 3.2.*

Proof. Consider θ a non-trivial, semi-basic 1-form, homogeneous of order $(p - 1)$ that satisfies the conditions (GH_1) and (GH_3) of Theorem 3.2. We will prove that $\delta_S(i_S\theta/p) = \sigma$. In this case, the Lagrangian is given by $L = i_S\theta/p$ and it is homogeneous of order p .

For θ , satisfying the condition (GH_1) , we use [3, Proposition 4.2]. It follows that

$$(6.1) \quad L = \frac{1}{p}i_S\theta$$

is the only p -homogeneous Lagrangian that satisfies $d_JL = \theta$.

We use the same argument that we used in the proof of Theorem 3.2, from formula (3.11) to formula (3.14). It follows that the condition (GH_3) implies that the 2-form $d_h\theta - d_J\sigma/2$ is a basic form and hence it is homogeneous of order 0. Since θ is homogeneous of order $(p - 1)$, and σ is homogeneous of order $p > 1$, it follows that $d_h\theta - d_J\sigma/2$ is homogeneous of order $p - 1 \neq 0$. Consequently, we obtain that this 2-form has to vanish and therefore we have

$$(6.2) \quad d_h\theta = \frac{1}{2}d_J\sigma.$$

In formula (6.2) we apply to the left the interior product i_S and use the commutation formula (5.2) to evaluate $i_Sd_h\theta$ and $i_Sd_J\sigma$. We have

$$-d_h i_S\theta + \mathcal{L}_S\theta + i_{[h,S]}\theta = \frac{1}{2}(-d_J i_S\sigma + \mathcal{L}_{\mathbb{C}}\sigma + i_{[J,S]}\sigma).$$

Using the fact that $d_JL = \theta$ it follows that $i_{[h,S]}\theta = i_{J \circ [h,S]}dL = -d_vL$. Now, we use the p -homogeneity of σ and $i_{[J,S]}\sigma = i_{h-v}\sigma = \sigma$ and hence we obtain

$$-pd_hL + \mathcal{L}_Sd_JL - d_vL = -\frac{1}{2}d_J i_S\sigma + \frac{p+1}{2}\sigma.$$

From the above formula we can express the Lagrange differential as follows

$$(6.3) \quad \delta_S L = (p-1)d_hL - \frac{1}{2}d_J i_S\sigma + \frac{p+1}{2}\sigma.$$

We will evaluate now the two 2-forms in both sides of formula (GH_3) on the pair of vectors (\mathbb{C}, S) . First we have $i_{\mathbb{C}}d_v\theta = i_{\mathbb{C}}d\theta = \mathcal{L}_{\mathbb{C}}\theta = (p-1)\theta$. From this formula and using also (6.1) we have

$$i_S i_{\mathbb{C}}d_v\theta = (p-1)i_S\theta = p(p-1)L.$$

According to [3, Proposition 3.6], in the homogeneous case, the dynamical covariant derivative ∇ commutes with the inner products i_S and $i_{\mathbb{C}}$. From the above formula, it follows that the value of the 2-form $\nabla d_v \theta$ on the pair of vectors (\mathbb{C}, S) is given by

$$(6.4) \quad i_S i_{\mathbb{C}} \nabla d_v \theta = p(p-1)S(L).$$

Since $(\mathbb{F} + J)(\mathbb{C}) = S$, it follows that $(i_{\mathbb{F}+J} d_J \sigma)(\mathbb{C}, S) = d_J \sigma(S, S) = 0$. Using the fact that σ is semi-basic and p -homogeneous, we have $i_{\mathbb{C}} d_v \sigma = i_{\mathbb{C}} d \sigma = \mathcal{L}_{\mathbb{C}} \sigma = p\sigma$. It follows that the value of the 2-form from the right hand side of (GH_3) on the pair of vectors (\mathbb{C}, S) is given by

$$(6.5) \quad i_S i_{\mathbb{C}} \left(d_v \sigma - \frac{1}{2} i_{\mathbb{F}+J} d_J \sigma \right) = p i_S \sigma.$$

From the two formulae (6.4) and (6.5) it follows that

$$(6.6) \quad (p-1)S(L) = i_S \sigma.$$

If we use now the commutation formula (2.1) for J and S we obtain $d_{[J,S]} L = d_J S(L) - \mathcal{L}_S d_J L$. In this formula, we replace $S(L)$ from (6.6) and use the fact that $[J, S] = h - v$. It follows

$$d_h L - d_v L = \frac{1}{p-1} d_J i_S \sigma - \mathcal{L}_S d_J L.$$

From the above formula we can express the Lagrange differential as follows

$$(6.7) \quad \delta_S L = \frac{1}{p-1} d_J i_S \sigma - 2d_h L.$$

From the two formulae (6.3) and (6.7) it follows that $\delta_S L = \sigma$ and hence the spray S is of Finslerian type with covariant force field σ . \square

When $\sigma = 0$, Theorem 6.3 reduces to [21, Lemma 3.4] and [23, Theorem 3.4], where it has been shown that the Helmholtz condition which involves the Jacobi endomorphism is a consequence of the other Helmholtz conditions.

From Theorem 6.3, in the particular case when $\sigma = 0$, we obtain the following corollary that provides a characterization of the Finsler metrizable problem in terms of a semi-basic, 1-homogeneous, 1-form.

Corollary 6.4. *A spray S is Finsler metrizable if and only if there exists a semi-basic 1-form $\theta \in \Lambda^1(T_0 M)$ that satisfies the following sets of conditions*

(FMA) $d\theta$ is a symplectic form, $i_S \theta > 0$;

(FMD) $d_J \theta = 0$, $\nabla d_v \theta = 0$, $\mathcal{L}_{\mathbb{C}} \theta = \theta$.

The first two differential conditions, (FMD), for Finsler metrizability, represent the two Helmholtz conditions (H_1) and (H_3) of Corollary 3.3.

6.2. The case $p = 1$. Next theorem gives a reformulation of the generalised Helmholtz conditions in the presence of a covariant force field σ , homogeneous of order 1. In this case, the semi-basic 1-form σ has to satisfy the condition $i_S \sigma = 0$, due to the following argument. Consider a spray S and a 1-homogeneous, semi-basic 1-form σ . We assume that there is a 1-homogeneous Lagrangian L such that $\delta_S L = \sigma$. If we apply i_S to both sides of this formula we obtain $S(\mathbb{C}(L) - L) = i_S \sigma$ and hence we necessarily should have $i_S \sigma = 0$.

Theorem 6.5. *Consider S a spray and a semi-basic 1-form $\sigma \in \Lambda^1(T_0 M)$, homogeneous of order 1 that satisfies the condition $i_S \sigma = 0$. The spray S is of Finslerian type with covariant force field σ if and only if there exists a non-trivial, 0-homogeneous semi-basic 1-form $\theta \in \Lambda^1(T_0 M)$ that satisfies one of the following two equivalent sets of conditions*

- i) $d_J\theta = 0$, $d_h\theta = \frac{1}{2}d_J\sigma$;
- ii) (GH_1) , (GH_2) , (GH_3) .

Proof. In view of Theorem 3.2, it remains to prove the equivalence of the two sets of conditions.

First we assume that there exists a non-trivial, 0-homogenous, semi-basic 1-form θ that satisfies i). Since $d_J\theta = 0$, using [3, Proposition 4.2] it follows that $L = i_S\theta$ is the only 1-homogeneous Lagrangian such that $\theta = d_JL$. The second condition i) can be written as $d_J(2d_hL + \sigma) = 0$, which implies that there exists a locally defined 2-homogeneous function f such that $2d_hL + \sigma = -d_Jf$. We consider now the 2-homogeneous function $g = f - S(L)$. It follows $\delta_S L = \sigma + d_Jg$. We apply i_S to both sides of the last formula and obtain $0 = \mathbb{C}(g) = 2g$. Therefore $\delta_S L = \sigma$, which in view of Theorem 3.2 implies that the conditions ii) are satisfied.

For the converse, assume that there exists a non-trivial, 0-homogenous, semi-basic 1-form θ that satisfies the three condition $(GH_1) - (GH_3)$. As we have seen in the proof of Theorem 3.2, the two conditions (GH_2) and (GH_3) imply that there exists a basic 1-form β such that formula (3.15) is satisfied. The 0-homogenous, semi-basic 1-form $\tilde{\theta} = \theta - \beta$ is non-trivial, satisfies the set of conditions i) and this completes the proof. \square

When $\sigma = 0$, the two conditions i) were used to characterize the projective metrizable of a spray in [5, Theorem 3.8]. Again, when $\sigma = 0$, the three conditions ii) were expressed in terms of the angular metric of some Finsler function as classic Helmholtz conditions in [10] and their equivalence with the set of conditions i) was proven in [10, Theorem 4].

In the particular case when the covariant force field σ is of gyroscopic type, we obtain, using Theorem 6.5, the following characterization for Finslerian gyroscopic sprays.

Corollary 6.6. *A spray S is of Finslerian gyroscopic type if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(T_0M)$ that satisfies the following conditions*

$$(6.8) \quad \mathcal{L}_C\theta = 0, \quad d_J\theta = 0, \quad d_h\theta = \omega,$$

for ω a basic 2-form.

Proof. If ω is a basic 2-form, using formula (5.3), it follows that $d_Ji_S\omega = 2\omega$. Therefore, the last condition i) of Theorem 6.5 is equivalent to the last condition (6.8). \square

As we have seen in the proof of Theorem 5.2, the last two conditions (6.8) are necessary conditions for an arbitrary semispray to be of gyroscopic type. Above Corollary 6.6 states that in the homogeneous case, these two conditions are also sufficient.

In the particular case when $\omega = 0$, the equations (6.8) coincide with the differential equations [5, (3.8)], which together with some algebraic equations, provide a characterization of projectively metrizable sprays.

7. EXAMPLES

7.1. Non-variational, projectively metrizable sprays that are dissipative. In this subsection we will provide examples of projectively metrizable sprays, which are not Finsler metrizable, and hence not variational but are of dissipative type.

A spray S is *projectively metrizable* if there exists a Finsler function F such that $\delta_SF = 0$. We underline the fact that F is a Finsler function and it is not a regular Lagrangian. However, for a Finsler function F , we have that $L = F^2$ is a regular Lagrangian. A spray S is *Finsler metrizable* if there exists a Finsler function F such that $\delta_SF^2 = 0$. In this case, we say that S is the geodesic spray of the Finsler function F .

In [6] it has been shown that for a given geodesic spray, its projective class contains infinitely many sprays that are not Finsler metrizable. More exactly, in [6, Theorem 5.1] it has been shown that if S_F is the geodesic spray of a Finsler function F , then there are infinitely many values of $\lambda \in \mathbb{R}$ such that the projectively related sprays $S = S_F - 2\lambda F\mathbb{C}$ are not Finsler metrizable. We consider such a spray S , which is projectively metrizable and hence satisfies $\delta_S F = 0$, and we will prove that the spray S is of dissipative type. Using the fact that S_F is the geodesic spray of F we have that $S_F(F) = 0$ and hence $S(F) = S_F(F) - 2\lambda F\mathbb{C}(F) = -2\lambda F^2$. Therefore

$$(7.1) \quad \delta_S F^2 = 2S(F)d_J F + 2F\delta_S F = 2S(F)d_J F = -4\lambda F^2 d_J F.$$

The semi-basic 1-form $\sigma = -4\lambda F^2 d_J F$ is d_J -closed and from formula (7.1) it follows that the spray S is of dissipative type.

Within the class of sprays that we have discussed above, we will consider now a concrete example in dimension 2. We consider the spray $S \in \mathfrak{X}(\mathbb{R}^2 \times \mathbb{R}^2)$, given by

$$(7.2) \quad S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - ((y^1)^2 + (y^2)^2) \frac{\partial}{\partial y^1} - 4y^1 y^2 \frac{\partial}{\partial y^2}.$$

In [1, Example 7.2] it has been shown that the system of second order ordinary differential equations corresponding to this spray is not variational. Since it is a 2-dimensional spray, it follows that S is projectively metrizable. According to Corollary 4.4, it follows that the sprays S is of dissipative type. We will show here directly, that the spray S is of dissipative type by using the characterization (D_1) of Theorem 4.2. We will prove that there exists a semi-basic 1-form $\theta = \theta_1(x, y)dx^1 + \theta_2(x, y)dx^2$ that satisfies the two conditions (D_1) , which can be written as follows

$$(7.3) \quad \frac{\partial \theta_1}{\partial y^2} = \frac{\partial \theta_2}{\partial y^1}, \quad \frac{\delta \theta_1}{\delta x^2} = \frac{\delta \theta_2}{\delta x^1}.$$

For the given spray (7.2), the coefficients of the canonical nonlinear connection are given by

$$N_1^1 = y^1, \quad N_2^1 = y^2, \quad N_1^2 = 2y^2, \quad N_2^2 = 2y^1.$$

Using first condition (7.3), the second condition (7.3) can be written as follows

$$(7.4) \quad \frac{\partial \theta_1}{\partial x^2} - \frac{\partial \theta_2}{\partial x^1} = y^2 \left(\frac{\partial \theta_1}{\partial y^1} - 2 \frac{\partial \theta_2}{\partial y^2} \right) + y^1 \frac{\partial \theta_1}{\partial y^2}.$$

It is easy to see that the PDE system (7.4) has solutions. For example $\theta_1(x^1, y^1)dx^1 + \theta_2(x^2, y^2)dx^2$ is a solution to the system (7.4) if and only if $\partial \theta_1 / \partial y^1 = 2\partial \theta_2 / \partial y^2$. A Riemannian solution to this system is provided by $\partial \theta_1 / \partial y^1 = 2\partial \theta_2 / \partial y^2 = 2$. Such a semi-basic 1-form θ , which is also homogeneous of order 1, is given by $\theta = 2y^1 dx^1 + y^2 dx^2$. Using formula (6.1), from the proof of Theorem 6.3, it follows that the corresponding Lagrangian can be obtained as $L = i_S \theta$,

$$(7.5) \quad L = \frac{1}{2} (2(y^1)^2 + (y^2)^2).$$

We know that system (7.2) is dissipative since we found a semi-basic 1-form θ that satisfies the conditions (D_1) of Theorem 5.2. Using the calculations we did in the proof of Theorem 5.2 (first implication), we obtain that the dissipative function is given by

$$(7.6) \quad \mathcal{D} = -\frac{2}{3}(y^1)^3 - 2y^1(y^2)^2.$$

One can also check directly that for the spray S , given by formula (7.2), the Lagrangian L , given by formula (7.5), and the dissipative function \mathcal{D} , given by formula (7.6), we have $\delta_S L = d_J \mathcal{D}$, which means that S is of dissipative type.

7.2. A class of gyroscopic semisprays. Historically, gyroscopic systems are systems of second order ordinary differential equations in \mathbb{R}^n , of the form

$$(7.7) \quad \frac{d^2 x^i}{dt^2} = A_j^i \frac{dx^j}{dt} + B_j^i x^j,$$

where A_j^i is a constant, skew symmetric matrix and B_j^i is a constant, symmetric matrix.

We will consider now a generalization of the above system, on some open domain $\Omega \subset \mathbb{R}^n$,

$$(7.8) \quad \frac{d^2 x^i}{dt^2} + 2N_j^i(x) \frac{dx^j}{dt} + V^i(x) = 0.$$

Consider g_{ij} a scalar product in \mathbb{R}^n . Using the conditions (G_1) of the Theorem 5.2, we will prove that the system (7.8) is of gyroscopic type, with the Lagrangian function $L(x, y) = g_{ij}y^i y^j / 2$, if and only if the functions $N_j^i(x)$ and $V^i(x)$ satisfy the following conditions

$$(7.9) \quad g_{ik}N_j^k + g_{jk}N_i^k = 0, \quad g_{ik} \frac{\partial V^k}{\partial x^j} - g_{jk} \frac{\partial V^k}{\partial x^i} = 0.$$

In the particular case when $g_{ij} = \delta_{ij}$ is the Euclidean metric, the functions N_j^i are constant and $V^i(x)$ are linear functions, the conditions (7.9) assure that (7.7) is a gyroscopic system.

We will use the local version (LG_1) of the set of conditions (G_1) of Theorem 5.2 to test when the system (7.8) is of gyroscopic type. For $L = g_{ij}y^i y^j / 2$, its Poincaré-Cartan 1-form is $\theta = dJL = g_{ij}y^j dx^i$. Therefore, we want to find the necessary and sufficient conditions for the existence of a basic 2-form $\omega \in \Lambda^2(\Omega)$ that satisfies the conditions (G_1) of Theorem 5.2. As we have seen in the proof of Theorem 5.2, the basic 2-form ω is necessarily given by $d_h \theta = \omega$. Locally, the components ω_{ij} of the 2-form ω are given by

$$(7.10) \quad \omega_{ij} = N_i^k g_{kj} - N_j^k g_{ki}.$$

We pay attention now to the second condition (LG_1) , which reads $\nabla g_{ij} = 0$ and implies

$$(7.11) \quad N_i^k g_{kj} + N_j^k g_{ki} = 0,$$

which is first condition (7.9). From the two formulae (7.10) and (7.11) it follows

$$(7.12) \quad N_j^i(x) = \frac{1}{2} g^{ik} \omega_{jk}(x).$$

We use formula (2.4) to compute the local components R_j^i of the Jacobi endomorphism,

$$R_j^k = 2 \frac{\partial N_l^k}{\partial x^j} y^l + \frac{\partial V^k}{\partial x^j} - \frac{\partial N_j^k}{\partial x^l} y^l - N_l^k N_j^l.$$

From this formula, and using formula (7.12), it follows

$$(7.13) \quad g_{ik} R_j^k - g_{kj} R_i^k = \left(\frac{\partial \omega_{ij}}{\partial x^l} + \frac{\partial \omega_{li}}{\partial x^j} + \frac{\partial \omega_{jl}}{\partial x^i} \right) y^l + g_{ik} \frac{\partial V^k}{\partial x^j} - g_{jk} \frac{\partial V^k}{\partial x^i}.$$

Therefore, last condition (LG_1) is satisfied if and only if the second condition (7.9) is satisfied. It follows that the system (7.8) is gyroscopic if and only if the two conditions (7.9) are satisfied.

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